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On the Real Cubic Fields

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Abstract. In this paper the authors announce a table of the 4753 totally real nonconjugate nonabelian cubic fields with discriminant less than 100000. Each field is given by its discriminant, the coefficients of a generating polynomial and the index of this polynomial over the field. A basis of the integers of the field is also given. Some differences with other tables are pointed out.

Godwin and Samet [3] have described the construction of a table of real cubic fields with discriminants $D < 20000$. Using similar methods Angell [1] extends these results up to $D < 100000$. Using a different method the authors [5] have constructed a table of the 4753 real nonabelian cubic fields with discriminants $D < 100000$. Here and in the sequel triplets of conjugate fields are counted once only.

The method developed in [5] generalizes the one used in [4] and the table constructed there gives, for each cubic field K, its discriminant, an irreducible polynomial $f(X)$ which defines K, the index of $f(X)$ over K and a basis for the integers of K.

In this work we present some consequences derived from [5]. In particular we have discovered that ten fields are missing from Angell's table. In Table 1 we give each of these ten fields and its corresponding class numbers h. The field K, with discriminant D, is generated by a root θ of the polynomial $f(X) = X^3 - AX + B$; S is the index of θ, and $\{1, \theta, \alpha\}$ is a basis of the integers of K, where

$$\alpha = \frac{\theta^2 + T\theta + (T^2 - A)}{S}.$$

TABLE 1

D	A	B	S	T	h
32404	64	194	1	0	1
35996	167	552	17	−4	1
37108	76	244	2	0	2
37133	167	374	20	−7	1
38905	73	147	5	1	1
39992	65	198	1	0	1
43165	163	482	16	−7	1
43173	30	49	1	0	1
43176	138	576	6	0	1
95484	183	936	3	0	1

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Table 1 allows the corrections of the results of [1], in particular the statistical table there given. However, as that table has some other mistakes, we prefer to reproduce it with its correct values in Table 2.

TABLE 2

		Class Number								
Bounds on D	No. of fields	1	2	3	4	5	6	7	8	9
1 to 10000	382	358	9	14	1	-	-	-	-	-
10001 to 20000	450	408	20	20	2	-	-	-	-	-
20001 to 30000	467	415	26	21	2	2	-	1	-	-
30001 to 40000	479	425	24	24	2	4	-	-	-	-
40001 to 50000	485	418	29	33	3	1	1	-	-	-
50001 to 60000	500	442	27	23	1	1	2	3	1	-
60001 to 70000	490	417	32	33	3	4	-	-	-	1
70001 to 80000	509	436	35	30	3	3	2	-	-	-
80001 to 90000	514	432	44	33	2	2	-	1	-	-
90001 to 100000	528	442	42	37	1	2	2	2	-	-

Let $N(x)$ be the number of nonabelian real cubic fields with discriminant $D < x$, $P(x)$ the number of real cubic fields (abelian and nonabelian) with discriminant $D < x$, and

$$c(x) = \frac{P(x)}{x}\pi^2.$$

In [2] it is proved that

$$\underline{\lim}_{x\to\infty} c(x) \geqslant 1/240 \quad \text{and} \quad \overline{\lim}_{x\to\infty} c(x) \leqslant 5/4,$$

and it is observed that both constants could be improved. In Table 3 we give the values of $N(x)$, $P(x)$ and $c(x)$ for several values of x.

TABLE 3

x	$N(x)$	$P(x)$	$c(x)$
10000	366	382	0.3770
20000	808	832	0.4106
30000	1270	1299	0.4274
40000	1746	1778	0.4387
50000	2227	2263	0.4467
60000	2725	2763	0.4545
70000	3211	3253	0.4587
80000	3716	3762	0.4641
90000	4229	4276	0.4689
100000	4753	4804	0.4741

We now give a sketch of the method employed (for details see [5]). Let $\bar{D} > 0$ be an integer. We propose to determine all conjugate triplets of noncyclic cubic fields with discriminant D such that $0 < D \leqslant \bar{D}$.

Each of these triplets is defined by a polynomial $f(A, B, X) = X^3 - AX + B$, where $A > 0$ and $B > 0$ are integers such that

(i) $f(A, B, X)$ is irreducible in $Q[X]$.

(ii) There is no integer $m > 1$ such that $m^2 \mid A$, $m^3 \mid B$, and the discriminant of $f(A, B, X)$ is

$$D(A, B) = 4A^3 - 27B^2 = DS^2,$$

where $S \geqslant 1$ is an integer.

We consider the congruences

$$A \equiv 3 \pmod 9, \qquad B \equiv \pm(A - 1) \pmod{27}. \tag{1}$$

By Voronoi's theorem, if the congruences (1) are not satisfied, then there are integers T, U, and V such that

(iii) $-S/2 < T \leqslant S/2$,

(iv) $3T^2 - A = US$,

(v) $T^3 - AT + B = VS^2$.

(vi) If we replace S by an integer $\bar{S} > S$ such that $\bar{S}^2 \mid D(A, B)$, then there are no integers T, U, V that satisfy the above-mentioned conditions.

If the congruences (1) are satisfied, then it follows that $S = 27S'$, and there are integers T, U, V such that

(vii) $-3S'/2 < T \leqslant 3S'/2$,

(viii) $3T^2 - A = 9US'$,

(ix) $T^3 - AT + B = 27VS'^2$.

(x) If we replace S' by an integer $\bar{S}' > S'$ such that $(27\bar{S}')^2 \mid D(A, B)$, then there are no integers T, U, V that satisfy the above conditions.

It follows that if we choose a minimal T, with each pair (A, B) is associated a unique quadruple (S, T, U, V) of integers that satisfies the above relations in each case. Also, each pair (A, B) determines a binary quadratic form $F(A, B)$ defined in such a way that if $K(A, B)$ is a cubic field defined by $f(A, B, X)$, the minimal polynomial of a nonzero integer γ of $K(A, B)$ with null trace is

$$X^3 - A'X - N(\gamma),$$

where A' is an integer represented by $F(A, B)$.

The coefficients of $F(A, B)$ are functions of S, T, U, V, and the discriminant of $F(A, B)$ is $-D/3$ if the congruences (1) are not satisfied and $27 \mid D$, and it is $-3D$ otherwise (see [5]).

Thus $F(A, B)$ is positive definite. Moreover, there is a pair (A', B'), which defines the same triplet of cubic fields as (A, B) if and only if A' is a nonzero integer represented by $F(A, B)$, and in such a case the quadratic forms $F(A, B)$ and $F(A', B')$ represent the same integers.

These considerations allow us to assume that A is the least integer represented by $F(A, B)$, and we obtain the bounds:

$$4 \leqslant A \leqslant \sqrt{\bar{D}} \quad \text{and} \quad 1 \leqslant B < 2\frac{A}{3}\sqrt{\frac{A}{3}}\,.$$

Using these bounds one can build up all pairs (A, B) each of which determines a triplet of conjugate noncyclic cubic fields with discriminant D such that $0 < D \leqslant \bar{D}$.

It can occur that two different pairs (A, B) and (A', B') determine the same triplet of cubic fields. This happens, as is proved in [5], if and only if $F(A, B)$ and $F(A', B')$ are equivalent quadratic forms. This provides us with an easy way to eliminate pairs which determine the same triplet of cubic fields.

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